Magnetic field induced phase transitions in spin ladders with ferromagnetic legs

T. Vekua,[1] G.I. Japaridze,[2] and H.-J. Mikeska[1]

[1] *Institut für Theoretische Physik, Universität Hannover, 30167 Hannover, Germany*

[2] *Andronikashvili Institute of Physics, Tamarashvili 6, 0177, Tbilisi, Georgia*

We study the phase diagram of spin ladders with ferromagnetic legs under the influence of a symmetry breaking magnetic field in the weak coupling effective field theory by bosonization. For antiferromagnetic interleg coupling we identify two phase transitions introduced by the external magnetic field. In order to establish the universality of the phases we supplement the bosonization approach by results from a strong coupling (rung) expansion and from spin wave analysis.

PACS numbers: 75.10.Jm Quantized spin models

I. INTRODUCTION

Recently there has been considerable interest in the study of magnetic field-induced effects in low-dimensional quantum spin systems, in particular devoted to the critical properties of spin $S = 1/2$ *isotropic antiferromagnetic* two-leg ladders in an external magnetic field. In parallel, there was remarkable progress in recent years in the fabrication of such ladder compounds[1]. Since antiferromagnetic two-leg ladder systems with $S = 1/2$ have a gap in the spin excitation spectrum, they reveal an extremely rich behavior, dominated by quantum effects, in the presence of a magnetic field. These quantum phase transitions were intensively investigated both theoretically, using different analytical and numerical techniques[8,9,10,11,12,13,14,15,16], and also experimentally[2,3,4,5,6,7].

Ladder models with *ferromagnetic legs* have been much less studied, although they exhibit many interesting aspects[17,18,19]. It is true that up to now no materials are available which realize these models. However, from the theoretical point of view these systems are extremely interesting, since they open up a new large class of systems for the study of complicated quantum behavior, unexpected in more conventional spin systems. The variety of possibilities is seen already from Fig. 1: here the ground state phase diagram[19] of a two-leg ferromagnetic ladder is presented in the variables intraleg exchange anisotropy (Δ) and (isotropic) interleg coupling ($J_\perp$). The ground state phase diagram contains, besides the fully gapped rung-singlet and Haldane phases (commonly known from the case of antiferromagnetic ladders[17,20]), the spin-liquid phase with easy-plane anisotropy ($XY1$), the ferromagnetic and the stripe-ferromagnetic phases which are realized only in the case of ferromagnetic legs ($0 \leq \Delta \leq 1$).

In this paper we study the effect of an external magnetic field on the phase diagram of this system. In particular, we focus our attention on the study of new field induced effects in the case of the easy-plane $XY1$ phase and in the rung-singlet phase in the vicinity of the single chain ferromagnetic instability point $\Delta = 1$.

The effect of the uniform magnetic field, applied parallel to the anisotropy (z) axis, on the ground state properties of the two-leg ladder systems is known from the investigation by Schulz[17] for the case of the corresponding spin $S = 1$ Heisenberg chain model: In the case of the gapless $XY1$ phase an external magnetic field leads to the appearance of a magnetization in z direction for arbitrary small magnetic fields. In the case of the gapped rung-singlet phase the magnetization appears only at a finite critical value of the magnetic field which is equal to the spin gap[8,17]. This behavior is generic for the spin gapped $U(1)$ symmetric systems in a magnetic field: The magnetic field leaves the in-plane rotational invariance unchanged[21] and the transition belongs to the universality class of the commensurate-incommensurate (C-IC) transitions[22,23].

On the other side, for the $U(1)$ symmetric phase such as the $XY1$ phase, the effect of a uniform *transverse field* is highly nontrivial. In the case of classical anisotropic spin chains this effect has been studied more than two decades ago[24]. However, in the case of the antiferromagnetic XXZ quantum chain this problem is still the subject of intensive recent studies[25,26,27,28].

In this paper, we study the effect of a *uniform transverse* magnetic field on the ground state phase diagram of a two-leg ladder with *anisotropic, ferromagnetically interacting* legs coupled by antiferromagnetic interleg exchange.

The outline of the paper is as follows: In section II we review the model and its bosonized version in the continuum limit. We discuss the phases and phase transitions emerging from the XY1 and rung-singlet phases in a transverse magnetic field in sections III (in the weak coupling approach) and IV (in the limit of strong rung exchange). We shortly summarize our results in section V. In appendix A we present the spin-wave approach to locate the ferromagnetic transition line starting from the saturated phase.

II. MODEL

Here we present a brief introduction to the model and in particular to its bosonized version in the continuum limit. The Hamiltonian we consider is given by:

$$H = H_{leg}^{(1)} + H_{leg}^{(2)} + H_\perp \,, \tag{1}$$

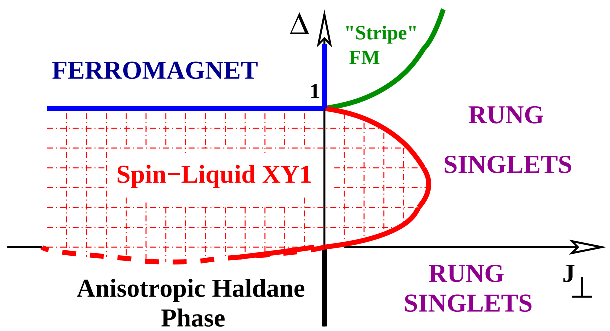

FIG. 1: Schematic picture of the ground state phase diagram of the two-leg ladder in the variables intraleg exchange anisotropy (Δ) and isotropic interleg coupling ($J_\perp$). The limiting case of *isotropic ferromagnetic* legs corresponds to $\Delta = 1$.

where the Hamiltonian for leg α is

$$H_{leg}^{(\alpha)} = - \; J\sum_{j=1}^{N}\Big(S^x_{\alpha,j}S^x_{\alpha,j+1} + S^y_{\alpha,j}S^y_{\alpha,j+1} + \;\Delta\, S^z_{\alpha,j}S^z_{\alpha,j+1}\Big) - h^{ext}\sum_{j=1}^{N} S^x_{\alpha,j}\,, \qquad (2)$$

and the interleg coupling is given by

$$H_\perp = J_\perp \sum_{j=1}^{N} \vec{S}_{j,1}\vec{S}_{j,2}\,. \qquad (3)$$

Here $S^{x,y,z}_{\alpha,j}$ are spin $S = 1/2$ operators on the j-th rung, and the index $\alpha = 1,2$ denotes the ladder legs. The intraleg coupling constant is ferromagnetic, $J > 0$, and therefore the limiting case of *isotropic ferromagnetic* legs corresponds to $\Delta = 1$. We will restrict ourselves to the case $0 \le \Delta \le 1$.

We use the following bosonization expressions for spin operators, adapted to the case of ferromagnetic exchange (for details see[19]):

$$\begin{aligned} S^x_{j,\alpha} &\simeq \frac{c}{\sqrt{2\pi}} :\cos\sqrt{\frac{\pi}{K}}\theta_\alpha: \\ &+ (-1)^j \frac{ib}{\sqrt{2\pi}} :\sin\sqrt{4\pi K}\phi_\alpha \sin\sqrt{\frac{\pi}{K}}\theta_\alpha:, \qquad (4)\\ S^y_{j,\alpha} &\simeq \frac{c}{\sqrt{2\pi}} :\sin\sqrt{\frac{\pi}{K}}\theta_\alpha: \\ &- (-1)^j \frac{ib}{\sqrt{2\pi}} :\sin\sqrt{4\pi K}\phi_\alpha \cos\sqrt{\frac{\pi}{K}}\theta_\alpha:, \qquad (5)\\ S^z_{j,\alpha} &= \sqrt{\frac{K}{\pi}}\partial_x\phi_\alpha + \\ &(-1)^j \frac{a}{\pi} :\sin\sqrt{4\pi K}\phi_\alpha(x):\,. \qquad (6) \end{aligned}$$

Here, $\phi(x)$ and $\theta(x)$ are dual bosonic fields, $\partial_t\phi = u\partial_x\theta$ and K is the Luttinger liquid parameter

$$K = \frac{\pi}{2\arccos\Delta}\,. \qquad (7)$$

Now we introduce the symmetric and antisymmetric combinations of the bosonic fields $\phi_\pm = \sqrt{1/2}\,(\phi_1 \pm \phi_2)$, $\theta_\pm = \sqrt{1/2}\,(\theta_1 \pm \theta_2)$ and after rescaling these fields we obtain finally as effective bosonic Hamiltonian

$$\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^- + \mathcal{H}^\pm\,, \qquad (8)$$

where

$$\begin{aligned} \mathcal{H}^+ &= \frac{u_+}{2}[(\partial_x\theta_+)^2 + (\partial_x\phi_+)^2] \\ &+ M_+\cos\sqrt{8\pi K_+}\phi_+(x)\,, \qquad (9)\\ \mathcal{H}^- &= \frac{u_-}{2}[(\partial_x\theta_-)^2 + (\partial_x\phi_-(x))^2] \\ &+ \mathcal{J}_\perp\cos\sqrt{\frac{2\pi}{K_-}}\theta_-(x)\,, \qquad (10)\\ \mathcal{H}^\pm &= -h\cos\sqrt{\frac{\pi}{2K_-}}\theta_-(x)\cos\sqrt{\frac{\pi}{2K_+}}\theta_+(x) \qquad (11) \end{aligned}$$

Here

$$K_\pm \simeq K\left(1 \mp \frac{J_\perp}{2\pi J}\frac{2K-1}{\sin(\pi/2K)}\right)\,, \qquad (12)$$

$\mathcal{J}_\perp \sim J_\perp$, $h \sim h^{ext}$ and $u_\pm$ are the velocities of the symmetric and antisymmetric modes.

In deriving (8), several terms which are strongly irrelevant in our case of a ladder with ferromagnetic legs and applied transverse magnetic field, were omitted. For details of the full Hamiltonian we refer the reader to[19].

We note that at $h^{ext} = 0$ the effective theory of the original ladder model is given by two decoupled quantum sine-Gordon models which describe, respectively, the symmetric and antisymmetric degrees of freedom. The infrared properties of the antisymmetric field are governed by the *strongly relevant* operator $\mathcal{J}_\perp\cos\sqrt{2\pi/K_-}\theta_-$ with the scaling dimension $(2K_-)^{-1} \le 1/2$. Therefore, the *antisymmetric sector is gapped at arbitrary* $J_\perp \neq 0$. Fluctuations of the field $\theta_-(x)$ are completely suppressed and the field θ_- gets ordered with expectation values

$$\langle\theta_-\rangle = \begin{cases} \sqrt{K_-\pi/2} & \text{at } \mathcal{J}_\perp > 0 \\ 0 & \text{at } \mathcal{J}_\perp < 0 \end{cases}\,. \qquad (13)$$

The infrared properties of the symmetric field are governed by the *marginal* operator $M_+\cos\sqrt{8\pi K_+}\phi_+$. As shown in[19], the symmetric mode remains gapless for ferromagnetic interleg exchange and arbitrary $0 \le \Delta \le 1$, while in the case of antiferromagnetic interleg exchange it is gapless in a finite regime in $\Delta, J_\perp$ parameter space given approximately by $\gamma_1 J_\perp/J \le \Delta \le 1 - \gamma_2 J_\perp/J$. Here γ_1 and γ_2 are positive constants (more rigorously smooth functions of the anisotropy) of the order of unity. Following Schulz[17] who has discussed a similar phase in the context of the spin $S = 1$ chain we denoted this phase as *spin liquid XY1* phase for the spin ladder. At $J_\perp > 0$, outside of this regime, the symmetric mode is also gapped and the field ϕ_+ gets ordered and pinned in

one of its possible minima. We denoted the fully gapped phase, realized in the case of antiferromagnetic interleg exchange as rung-singlet phase. In this phase spins on the same rung tend to form a singlet and the ground state corresponds to the state with a singlet pair on each rung. The ground state phase diagram of the system at $h^{ext} = 0$ is given schematically in Fig. 1.

Below we study the ground state phase diagram of the two-leg ladder with *anisotropic ferromagnetic* legs in the presence of a magnetic field.

III. THE SPIN LADDER IN THE PRESENCE OF AN IN-PLANE MAGNETIC FIELD

In this Section we consider the effect of a transverse magnetic field on the ground state phase diagram of the spin-liquid $XY1$ phase. Since the magnetic field breaks the in-plane rotational symmetry of the $XY1$ phase, it is clear that it will eliminate the gapless $XY1$ phase and that the system will acquire a gap in the excitation spectrum for arbitrary small h^{ext}. Both fields (symmetric and antisymmetric) will be pinned in their respective minima. In this case the very first direct approach to study the ground state phase diagram of the model is to use the quasiclassical Ginzburg-Landau type analysis. This approach does not cover the limiting case $\Delta = 1$ where bosonization does not apply. The $\Delta = 1$ case will be discussed in section IV.

A. Quasiclassical Ginzburg-Landau analysis

In the presence of an in-plane magnetic field the effective Hamiltonian (8) reduces to two free Gaussian fields coupled by the following effective potential

$$\mathcal{U}_{eff}\left(\Theta_{-},\Theta_{+}\right) = \mathcal{J}_{\perp}\cos 2\Theta_{-} - h\cos\Theta_{-}\cos\Theta_{+}\,, \quad (14)$$

where $\Theta_{\pm} = \sqrt{\pi/2K_{\pm}}\theta_{\pm}$. To find the vacuum expectation values of the pinned field we search for the minima of the effective potential $\mathcal{U}_{eff}$ with respect to Θ_- and Θ_+. The straightforward analysis gives the following sets of vacua:

- $h > h_c = 4\mathcal{J}_{\perp}$

$$\begin{aligned} \text{I.}\quad \Theta_{+} &= \pi\,,\ \Theta_{-} = \pi\,,\ \ (\mathrm{mod}\ 2\pi) \\ \text{II.}\quad \Theta_{+} &= 0\,,\ \Theta_{-} = 0\,,\ \ (\mathrm{mod}\ 2\pi) \end{aligned} \quad (15)$$

- $h < h_c = 4\mathcal{J}_{\perp}$

$$\begin{aligned} \text{I.}\quad \Theta_{+} &= \pi\ \Theta_{-} = \pi \pm \vartheta_0\,,\ \ (\mathrm{mod}\ 2\pi) \\ \text{II.}\quad \Theta_{+} &= 0,\ \Theta_{-} = \pm\vartheta_0\,,\ \ (\mathrm{mod}\ 2\pi) \end{aligned} \quad (16)$$

where

$$\vartheta_0 = \arccos(h/4\mathcal{J}_{\perp}).$$

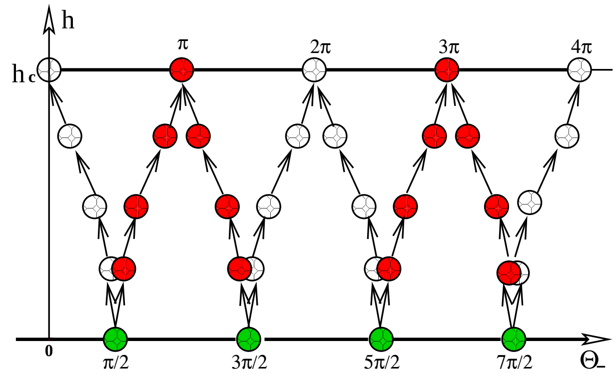

FIG. 2: Evolution of the set of minima of the antisymmetric field with increase of the applied transverse magnetic field.

At $h = 0$ only the antisymmetric field is gapped and its set of available vacua is given by $\Theta_- = \pi/2, (\mathrm{mod}\ \pi)$. At $h \neq 0$ the symmetric field gets pinned and the set of possible vacua of the symmetric field does not change with field. On the other hand, at arbitrary $0 < h < 4\mathcal{J}_{\perp}$, each minimum in the antisymmetric sector splits into two degenerate minima (see Fig. 2). At $h \to h_c$, $\vartheta_0 \to \pi/2$ the split minima join each other and form a new set of possible vacua for the antisymmetric field which is fixed for arbitrary $h > h_c$. At the critical point the effective potential transforms into the $(\Theta_-)^4$ potential, which is common in describing the Ising universality class. Since the location of the minima in the symmetric sector does not change, we conclude that the transition involves only the antisymmetric sector.

An analogous quasiclassical analysis was carried out by Fabrizio et al.[29] in connection with the ionic Hubbard model. There two transitions were identified for finite values of alternating on-site energies. We note as an important difference that assuming a vanishing order parameter along the direction of an infinitesimally small applied magnetic field would be self-contradictory in our case, because here no appropriate marginal operators are present.

Note that in the case of ferromagnetic interleg exchange ($J_\perp < 0$), the set of minima of the effective potential for arbitrary $h^{ext} \neq 0$ is given by the set (15). Therefore, no transitions with increasing field are expected in this case.

B. Phase diagram

The quasi-classical analysis performed above allows to sketch qualitatively the ground state phase diagram of the model under consideration.

As soon as we switch on an infinitesimally small magnetic field in X direction, breaking the in-plane rotation symmetry, the $XY1$ phase is expelled, the system becomes gapped and acquires a finite order parameter. Using the bosonized expressions for the smooth parts of

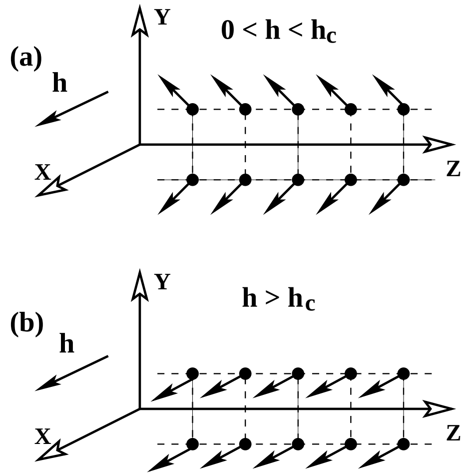

FIG. 3: Crossover from stripe-ferromagnetic (a) to ferromagnetic (b) states.

the spin operators Eqs. (4)-(5), expressed in terms of the symmetric and antisymmetric fields and the vacuum expectation values of the corresponding fields (15)-(16) one easily obtains that at $0 < h < h_c$ the spin ladder acquires

- uniform magnetization in the direction of applied field M_x

$$\begin{aligned} M_x &= \left\langle S^x_{j,1}(x)\right\rangle = \left\langle S^x_{j,2}(x)\right\rangle \\ &\simeq \left\langle \cos\Theta_+\right\rangle \left\langle \cos\Theta_-\right\rangle = h/4\mathcal{J}_\perp \end{aligned} \quad (17)$$

- opposite magnetization of legs in the in-plane direction perpendicular to the field:

$$\begin{aligned} M_y &= \left\langle S^y_{j,1}(x)\right\rangle = -\left\langle S^y_{j,2}(x)\right\rangle \\ &\simeq \left\langle \cos\Theta_+\right\rangle \left\langle \sin\Theta_-\right\rangle = \sqrt{1-(h/4\mathcal{J}_\perp)^2}\,. \end{aligned} \quad (18)$$

This phase we denote as the "stripe-ferromagnetic" phase.

When the magnetic field exceeds the critical value $h > h_c$, the new set of vacua (15) is reached and the system passes into the ferromagnetic phase, where

$$M_x \simeq \left\langle \cos\Theta_-\right\rangle = 1 \text{ and } \quad M_y \simeq \left\langle \sin\Theta_-\right\rangle = 0\,. \quad (19)$$

Thus, the quasiclassical analysis of the ground state phase diagram of the $XY1$ phase in the presence of a transverse magnetic field shows a *phase transition* from the stripe-ferromagnetic to the ferromagnetic phase (we denote by 'ferromagnetic phase' the phase with the magnetization parallel to the external field as only nonvanishing order parameter). This analysis also indicates, that the transition belongs to the Ising universality class.

From the above it is also clear that in the case of ferromagnetic interleg exchange ($\mathcal{J}_\perp < 0$) the ferromagnetic phase is realized for arbitrary $h \neq 0$: This is so, since in this case the vacua of two terms in (14) do not exclude each other.

C. Double sine-Gordon analysis

According to the quasiclassical analysis the magnetic field induced transition in the $XY1$ phase from the partly polarized stripe-ferromagnetic phase into the ferromagnetic phase takes place along the critical line $h_c = 4\mathcal{J}_\perp$ and belongs to the Ising universality class. In this subsection we investigate the stability against quantum fluctuations of the results obtained so far. For this purpose we decouple the initial theory of interacting quantum multi-frequency sine-Gordon fields (8) to get

$$\begin{aligned} \mathcal{H}_{MF} &= \mathcal{H}^+ + \mathcal{H}^-\,, && (20)\\ \mathcal{H}^+ &= \frac{u_+}{2}[(\partial_x\theta_+)^2 + (\partial_x\phi_+)^2] \\ &- h_+\cos\beta_+\theta_+(x)\,, && (21)\\ \mathcal{H}^- &= \frac{u_-}{2}[(\partial_x\theta_-)^2 + (\partial_x\phi_-(x))^2] \\ &- h_-\cos\beta_-\theta_-(x) + \mathcal{J}_\perp\cos 2\beta_-\theta_-(x)\,, && (22) \end{aligned}$$

where

$$\beta_\pm = \sqrt{\pi/2K_\pm} \quad (23)$$

and

$$h_\pm = h\langle\cos\sqrt{\pi/2K_+}\theta_\mp\rangle \quad (24)$$

Thus the symmetric sector is described by the ordinary sine-Gordon theory with $\beta_+^2 \leq \pi/2$ for $0 \leq \Delta \leq 1$, and therefore with a strongly relevant massive term. On the other hand, in the antisymmetric sector we arrive at the double frequency sine-Gordon model with $\beta_-^2 \leq \pi/2$ and therefore strongly relevant basic and double-field operators.

In the double frequency sine-Gordon model the quantum phase transition in the ground state takes place when the vacuum configurations of the two *cosines* compete with each other[30], corresponding to the crossover from double well to single well potential in quasiclassical analysis. One easily verifies that this criterion can be applied analogously to the case of antiferromagnetic rung exchange ($J_\perp > 0$). The dimensional arguments based on equating physical masses produced by the two cosine terms separately is usually used to define the critical line:

$$\begin{cases} m_h &= (h)^{1/(2-dim[h])} \\ m_{J_\perp} &= J_\perp^{1/(2-dim[J_\perp])} \end{cases} \quad (25)$$

Equating these two masses we obtain the following expression for the critical line external magnetic field vs. interchain coupling:

$$h_{cr} = \eta J_\perp^{\mu(\Delta)}\,. \quad (26)$$

where

$$\mu = \frac{2-dim[h]}{2-dim[J_\perp]} = (8K^2-K)/2K_+(4K_- - 1) \quad (27)$$

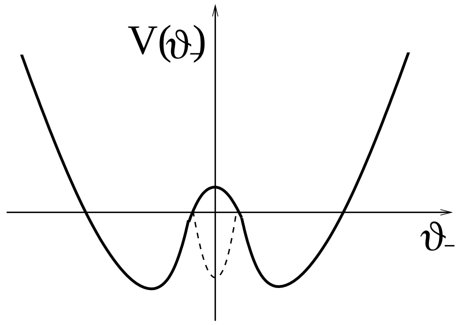

FIG. 4: Appearance of an additional minimum (dashed line) which could change the second order phase transition to first order, depending on the sign of the higher order harmonic generated.

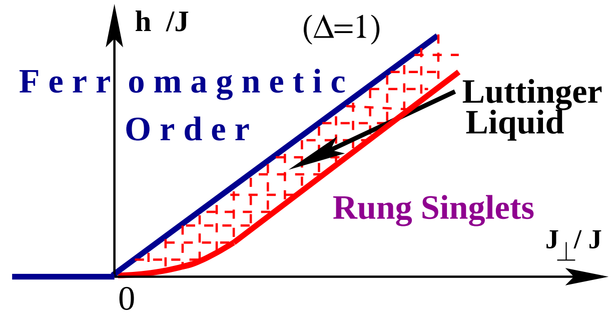

FIG. 5: Phase diagram of a spin ladder with the SU(2) symmetric ferromagnetic legs in a uniform magnetic field.

and η is some numerical constant of the order of unity. The self consistency of the mean field separation insures that (27) follows from equating masses produced by magnetic field and interchain couplings separately, before the mean field separation. From (27) we see $h_{cr} \simeq J_\perp$ in the limit $K \to \infty$, i.e. for the single chain ferromagnetic instability point. This will be seen to be consistent with the large rung coupling as well as with spin wave analysis (see respectively eqs. (34) and (A3)).

Finally we want to mention that the operator product expansion of the two lowest frequencies of the sine-Gordon theory for the range of anisotropy parameter $0 \le \Delta \le 1$ does not close, i.e. higher relevant harmonics are generated in the RG procedure. In some situations it is known that higher harmonics can destabilize second order phase transitions and make it weakly first order[31]. This happens when generated harmonics introduce new minima (e.g. the dashed minimum in Fig. (4)) which at the phase transition point coexists with the minima before the transition.

For example in the case of only the third harmonic retained, the position of the new minimum depends on the sign of the generated harmonics. We have followed the sign of the generated harmonic along the one loop RG flows and checked that in our case there is no indication of an upsetting of the second order phase transition. The new minima get pinned in between the minima of the first two harmonics; they thus may shift the critical line (26) to larger values of the magnetic field but otherwise do not introduce any qualitatively new effects.

IV. LARGE RUNG COUPLING RESULTS

In this section we consider the effect of an applied magnetic field on the ground state phase diagram of the two-leg ladder system with ferromagnetic legs (1) in the limiting case of strong rung coupling $J_\perp \gg J$. In this limit it is convenient to discuss the model by representing the *site*-spin algebra in terms of *on-bond*-spin operators[32]. In particular, in the case of isotropic interleg exchange the ladder can be mapped onto a single $S = 1/2$ chain[3,10,33]. For completeness we briefly discuss the mapping here: A given rung may be in the singlet or in the triplet state with energies given by

$$E_{t,\pm} = \frac{J_\perp}{4} \pm h^{ext}, \; E_{t,0} = \frac{J_\perp}{4}, \; E_s = -\frac{3J_\perp}{4}.$$

At $h^{ext} \le J_\perp$, one component of the triplet becomes closer to the singlet ground state such that for a sufficiently strong magnetic field we have a situation where the singlet and the $S_z = +1$ component of the triplet form a new effective spin 1/2 system. One can easily project the original ladder Hamiltonian (1) on the new singlet-triplet subspace

$$\begin{aligned} |\Uparrow\rangle &\equiv |t^+\rangle = |\uparrow\uparrow\rangle \\ |\Downarrow\rangle &\equiv |s\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \end{aligned} \tag{28}$$

This leads to the definition of the effective spin 1/2 operators

$$S^+_{n,\alpha=1,2} = (-1)^\alpha \frac{1}{\sqrt{2}} \tau^+_n \tag{29}$$

$$S^z_{n,\alpha=1,2} = \frac{1}{4}[I + 2\tau^z_n] \tag{30}$$

When expressed in terms of the effective spin operators (29)-(30), the original Hamiltonian (1) becomes

$$\begin{aligned} H_{\text{eff}} &= -J\sum_i [\frac{1}{2}\tau^z_i\tau^z_{i+1} + \tau^y_i\tau^y_{i+1} + \Delta\tau^x_i\tau^x_{i+1}] \\ &\quad -h_{eff}\sum_i \tau^z_i + Constant, \end{aligned} \tag{31}$$

where the effective magnetic field

$$h_{eff} = h^{ext} - J_\perp - \frac{1}{2}J. \tag{32}$$

Note that in writing (31) for convenience we have exchanged X and Z axis in effective spin space.

The Hamiltonian (31) describes a spin 1/2 *fully anisotropic* XYZ chain in an effective magnetic field.

There exists a special case, $\Delta = 1$, which allows for rigorous analysis. In this case the effective problem reduces to the theory of the XXZ chain with a *fixed ferromagnetic* XY anisotropy of $1/2$ in an effective magnetic field h_{eff}. The gapped phase at $h_{eff} < h_{eff}^{c1}$ for the ladder corresponds to the fully saturated magnetization phase for the effective spin chain pointing in the direction opposite to the applied field, whereas the massless phase for the ladder corresponds to the finite magnetization phase of the effective spin-1/2 chain[3]. The critical field h_{eff}^{c2} where the ladder is fully magnetized corresponds to the fully magnetized phase of the effective spin chain pointing along the direction of applied field. From the exact ground state phase diagram of the anisotropic XXZ chain in a magnetic field[34] using (32) we get that the *isotropic ferromagnetic ladder* in a magnetic field shows two second order phase transitions: at

$$h^{ext}{}_{c1} = J_\perp - J \tag{33}$$

a transition occurs from the rung dimer to a Luttinger liquid phase and at

$$h^{ext}{}_{c2} = J_\perp \tag{34}$$

a transition from a Luttinger liquid phase into the fully polarized phase.

The transition from the rung-singlet phase into the Luttinger liquid phase in the case of the *isotropic antiferromagnetic ladder* was studied in detail in several recent publications[10,11,12,33]. It was shown that in the case of the gapped rung-singlet phase the magnetization appears only at a finite critical value of the magnetic field equal to the spin gap. Since this behavior is generic for isotropic systems with spin gap[21], and the gap in the ladder system is governed by $J_\perp$ we conclude that the rung-singlet to Luttinger liquid phase transition line smoothly reaches zero at $J_\perp \to 0$ (see Fig.5).

Note that this large rung coupling analysis reveals that the phase transitions in the antiferromagnetically coupled ladder with ferromagnetic legs in uniform magnetic field are connected with those in a ladder with antiferromagnetic legs but in staggered magnetic field[16]. In both cases the magnetic field tries to promote triplets on rungs, while the antiferromagnetically coupled ladder supports on-rung singlets.

Away from the isotropic point $\Delta = 1$ the effective Hamiltonian (31) describes the fully anisotropic ferromagnetic XYZ chain in a magnetic field that is directed perpendicular to the easy axes. For the particular value of magnetic field $\mathrm{h}_{eff} = 0$, the effective XYZ chain is long range ordered in $Y-$direction, corresponding to the original ladder system being ordered in the direction perpendicular to the applied magnetic field with opposite magnetization on the legs (stripe-ferromagnetic phase). For larger values of the effective field it is clear that this striped ferromagnetic order will be replaced either by the rung singlet phase or the phase with only one order parameter - magnetization along the applied field. Thus we obtain the result that the phases appearing in large rung coupling and in weak rung coupling are identical. On general grounds one expects an Ising transition to take place at critical strengths of the magnetic field also for large couplings. Thus for $\Delta \neq 1$ the Luttinger liquid phase will be replaced by the striped ferromagnetic phase (as in the weak coupling limit), and the transitions become of Ising type due to the reduced symmetry (Fig. 6).

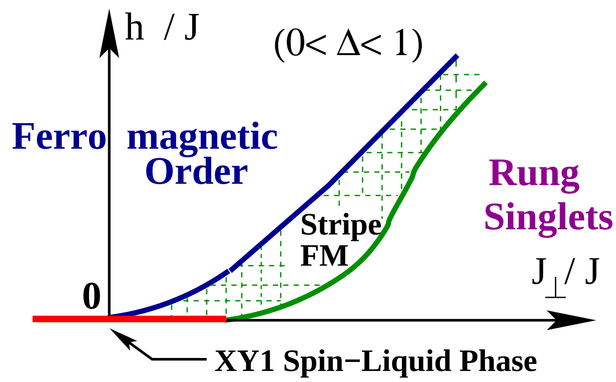

FIG. 6: Phase diagram of a ladder with anisotropic ferromagnetic legs in transverse magnetic field.

In order to determine the nature of the transition from rung singlets to the striped ferromagnetic state in the weak coupling limit we can use the fact that in the rung singlet phase the operator

$$J_\perp \cos\sqrt{8\pi K_+}\phi_+(x)$$

is relevant and the ground state consists of nonmagnetic singlets situated along the rungs of the ladder. On the other hand, while the in-plane magnetic field couples to the dual fields (disorder operators) we expect an Ising phase transition to take place with the appearance of the magnetization perpendicular to the applied field (stripe-FM) as dominant order parameter.

V. CONCLUSIONS

We have investigated the phase diagram of the $S = 1/2$ ladder with ferromagnetic legs under the influence of a uniform magnetic field breaking the in-plane rotational symmetry. In case of antiferromagnetic coupling between legs we identified two phase transitions in the plane of magnetic field vs interchain coupling. We have extended our analysis to the strong rung coupling limit and have identified a Luttinger liquid phase which replaces the stripe- ferromagnetic phase in the case of $SU(2)$ symmetric legs. The phase transition line in the case of $SU(2)$ ferromagnetic legs was confirmed also by the spin wave calculation starting from the saturated limit.

VI. ACKNOWLEDGEMENTS

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APPENDIX A: FERROMAGNETIC INSTABILITY

In this appendix we use the spin wave approach to determine the critical line corresponding to the *ferromagnetic instability* for the case of $SU(2)$ symmetric legs. We refer the reader to the appendix of (19) where we have considered in detail the analogous analysis of a ladder without magnetic field. In the case of $SU(2)$ symmetric legs it is straightforward to add the magnetic field term, since it couples to the diagonal operator. In this case the two sets of spin wave excitation frequencies[19] read:

$$\omega^-(q) = h^{ext} - \frac{J_\perp}{2} - J\cos q - \frac{J_\perp}{2} \tag{A1}$$

$$\omega^+(q) = h^{ext} - \frac{J_\perp}{2} - J\cos q + \frac{J_\perp}{2}. \tag{A2}$$

For $J_\perp > 0$ we have

$$\omega^-(q) < \omega^+(q)$$

and from the instability condition $\omega^-(q=0) = 0$ we obtain

$$h^{ext} = J_\perp \tag{A3}$$

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